Modeling Robot Swarms Using Integrals of Birth-Death Processes

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This article investigates the use of the integral of linear birth-death processes in the context of analyzing swarm robotics systems. We show that when a robot swarm can be modeled as a linear birth-death process, well-established results can be used to compute the expected value and/or the distribution of important swarm performance measures, such as the swarm activity time or the swarm energy consumption. We also show how the linear birth-death model can be used to estimate the long-term value of such performance measures and design robot controllers that satisfy constraints on these measures.

1. INTRODUCTION

Swarm robotics studies how to design large groups—swarms—of robots that cooperate to carry out tasks that are beyond the capabilities of the single constituent robots and that do so without any form of centralized control and without relying on external infrastructure.

One of the open problems in swarm robotics is how to design and implement robot swarms with a given guaranteed level of performance. In this article, we use linear birth-death processes to model robot swarms whose components—robots—have a limited lifetime. We show how this modeling approach can be used to design robot swarms with guaranteed levels of performance and to compute long-term swarm-level performance measures.

A birth-death process is a continuous-time Markov process where state transitions are either births or deaths—births increase the state variable by one, whereas deaths decrease it by one. Integrals of nonnegative stochastic processes arise frequently in engineering applications. Examples include the estimation of the expected energy

consumption by a system subject to random dynamics or the estimation of the cost associated to the execution of a task whose subtasks have random duration or cost. Other applications are found in queuing, storage, and inventory systems [Moran 1959]. In biology, integrals of nonnegative stochastic processes are often associated with the expected resource consumption of groups of animals [Mangel and Tier 1994], with the expected amount of toxins produced by bacteria [Puri 1967], or with the total cost of epidemics [Jerwood 1970; Gani and Jerwood 1972]. Results in the context of potential theory allow for the characterization of certain probabilistic properties of this type of integral. To the best of our knowledge, integrals of birth-death processes have not been investigated in the swarm robotics literature.

In swarm robotics, integrals of birth-death processes can be used to study properties of the system that depend on the number of robots in a certain state when the rate at which this number varies is known or measurable. For example, one could use the integral of a birth-death process to estimate the expected duration of an activity or the total energy consumption of a swarm, as these are properties that generally are a function of the number of active robots at each timestep. The approach presented in this article can be used to predict the performance[1] of the robot swarm over time and/or to tune the system parameters during the design phase to achieve a desired robot swarm performance.

The rest of the article is organized as follows. Section 2 reviews related work in mathematical modeling of swarm robotics systems and provides an overview of the literature on integrals of nonnegative stochastic processes. Section 3 explains how a robot swarm can be modeled as a linear birth-death process and introduces the basic notations and definitions. Section 4 gives a closed-form characterization of the integral of the linear birth-death process. In Section 5, we consider a cleaning task as an application scenario. We define a set of swarm performance measure and show how the presented model can be used to predict the long-term performance of the swarm and design a robot form that has a desired expected performance. Section 6 concludes the article.

2. OVERVIEW OF RELATED WORK

The article discusses the mathematical characterization of so-called swarm robotics functions using the integrals of birth-death processes. Therefore, we organize the related work into two parts. Section 2.1 discusses works that have investigated the mathematical modeling of swarm robotics systems. Section 2.2 presents an overview of the integrals of birth-death processes.

2.1. Mathematical Modeling of Swarm Robotics Systems

Since mathematical modeling of multirobot systems is a complex issue, roboticists have often opted for simulations and experiments to guide their research (see Brambilla et al. [2013] for an extensive review of recent work in swarm robotics). The complexity arising from the interactions of the individuals, the heterogeneity of the environment, and the drastic simplifications required in mathematical models are among the most reported obstacles in the way of a rigorous analysis of swarm systems.

Several mathematical models have been nonetheless proposed in the literature. A first type of model is based on so-called rate equations: a series of coupled differential equations describing the dynamics of the swarm. Examples are the characterization of collective behaviors such as aggregation, dispersion, and foraging [Martinoli et al. 1999, 2004; Lerman et al. 2001; Campo and Dorigo 2007; Winfield et al. 2008; Liu and

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[1]Examples of swarm performance measures include the following: the total activity time of the robot swarm, the total energy consumption of the robot swarm, or the total amount of work carried out by the robot swarm in the presence of spatial interferences.

Winfield 2010]. These techniques are used to model the average number of robots in each of the system states.

A second type of model is spatial models. Examples include the modeling approach of Hamann [2010], in which Brownian motion and Fokker-Planck equations are used for the study of space-time continuous models in simple swarm robotics systems. As such models are specialized toward the modeling of spatial aspects, they are not relevant for the work presented in this article.

A third type of model based on birth-death processes has been used to determine the expected time for the robots to cluster [Hereford 2010], to analyze the stability of robotics controllers for the distributed tracking problem [Shaw and Klavins 2008], and for the modeling of distributed robot deployment [Mather and Hsieh 2011]. In these works, probabilistic considerations are derived via ordinary differential equation (ODE) models, whereas in our work, the analysis is done using the integral of a function of the birth-death process.

Finally, a variety of specialized models for different swarm behaviors have been proposed. Winfield and Nembrini [2006] presented a formal method using temporal logic to specify emergent swarm behaviors. Soysal and Şahin [2006] derived an analytical model for aggregation using techniques from combinatorics and linear algebra. Varghese and McKee [2010] proposed a macroscopic model that tackles the problem of achieving pattern formation in swarm robotics using a complex plane representation, and Berman [2010] introduced an approach that uses analogies from the domain of chemical reaction modeling and applied it to swarm task-allocation problems. Quantitative analyses of important properties of swarm robotics systems, as well as the modeling and validation of complex scenarios using formal languages, were discussed in the work of Gjondrekaj et al. [2012] and references therein. Other works have applied mathematical modeling to decision-making problems in swarm robotics, such as Valenti and Hamann [2015] and Hamann et al. [2014]. Mathematical modeling was also used to tackle the problem of task allocation in robot swarms for time-constrained tasks [Khaluf et al. 2013, 2014].

The preceding modeling techniques neither allow one to deal with functions of the number of robots nor the integral over time of these functions. On the contrary, this is possible with the technique proposed in this article, which allows the modeling of several measures of swarm performance, as discussed later in Section 5.2, as well as the frequent situation in which robots have a limited operation time, such as that due to limited battery life or faults.

2.2. Brief Review of the Literature on Integrals of Markov Birth-Death Processes

Due to its relevance in numerous practical applications, the problem of evaluating integrals of Markov processes has been extensively studied outside the swarm robotics literature. An early result introduced in Hernández-Suárez and Castillo-Chavez [1999] presents a methodology for evaluating the expected value of the integral of a birth-death process, as well as the expected time to extinction for different kinds of processes. Pollett [2003] and Pollett and Stefanov [2002] have generalized the approach described in Hernández-Suárez and Castillo-Chavez [1999] to a wider variety of models and removed restrictive assumptions. In the case of birth-death processes, the Laplace transform of the distributions of first passage times and other important characteristics have been obtained [Ball and Stefanov 2001; Flajolet and Guillemin 2000; Pollett and Stefanov 2002]. Crawford and Suchard [2012] proposed an efficient error-controlled algorithm for computing transition probabilities for general birth-death processes. This promising method constitutes a robust computational tool to obtain finite-time transition probabilities for birth-death processes that are linked to the expected value of the process integral over arbitrary functions.
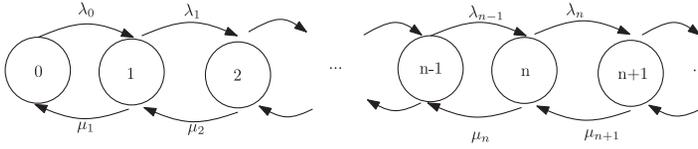
Fig. 1.   State diagram of a birth-death processes with rates λ_n and μ_n.

3. MODELING ROBOT SWARMS USING LINEAR BIRTH-DEATH PROCESSES

In this section, we analyze the integrals of swarm robotics functions—that is, the integrals of functions of birth-death processes that model swarm robotics systems. These integrals are computed over the time interval during which the swarm is active—the swarm activity time. A general birth-death process is a continuous-time Markov process $\mathcal{X} = \{X(t), t \geq 0\}$ that counts the number of arbitrarily defined elements in existence at time $t \geq 0$. The process is characterized by nonnegative instantaneous birth rates λ_n, $n \geq 0$, and death rates μ_n, $n \geq 1$. We consider swarm systems in which these rates are linear functions of the number n of elements (i.e., robots). The instantaneous transition of the process occurs from state $X(t) = n$ to either state $X(t') = n - 1$ or $X(t') = n + 1$.

The activation (birth) and deactivation (death) of the robots in the swarm happens as follows. At time $t = 0$, a large number of robots are gathered in a region called *nest*. An initial number of these robots is engaged in a certain task and is classified as active. Over time, active robots may independently stop the execution of the task that they are performing (i.e., deactivate) for several possible reasons, such as a fault in the robot, a low battery level, or an internal decision. This happens with rate μ_n, where n is the number of active robots. We assume that μ_n is known or measurable. Similarly, robots in the nest become active at rate λ_n.

Because we are considering linear birth-death processes, we have $\mu_n = n\mu$ and $\lambda_n = n\lambda$, where μ and λ are parameters of the system. As in this article we are interested in birth-death processes that lead the system to extinction—that is, the state in which all robots are inactive—we assume that $\lambda < \mu$.

The number of active robots at time t can be modeled as a linear birth-death process $X(t)$ on the state space $\{0, 1, \ldots\}$ with transition rates:[2]

$$\mu_n = n\mu, \quad \text{when} \quad n \geq 1, \tag{1}$$

$$\lambda_n = n\lambda, \quad \text{when} \quad n \geq 0. \tag{2}$$

Figure 1 shows the state diagram of the resulting birth-death process. In this article, the birth-death process $X(t)$ refers to the number n of active robots at time t, and the function $f(X(t))$ can be mapped to any function $f(n)$ of the number n of robots for which we are interested in computing the integral over time.

Figure 2 illustrates a possible realization of the process over a positive function $f(n)$, $n \in \{0, 1, \ldots\}$. For the sake of visualization, the function is represented as continuous. The values of the function $f(n)$ over the states of the chain are represented by levels of grey and can be thought of, for example, as the amount of energy consumed by a swarm of size n (with white being higher values). The integral of the resulting process can be associated to the amount of energy consumed by the swarm from time $t = 0$ until the complete deactivation of all robots. Similarly, depending on the meaning of the function $f(n)$, other properties, such as the average number of objects retrieved in a

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[2]From these equations, it is possible to compute the probabilities $P_{n,n+1}$ and $P_{n,n-1}$ of a unit increase or decrease in the number n of active robots [Ross 2006]: $P_{n,n+1} = \lambda n/(\lambda n + \mu n) = \lambda/(\lambda + \mu)$ and $P_{n,n-1} = \mu n/(\lambda n + \mu n) = \mu/(\lambda + \mu)$.
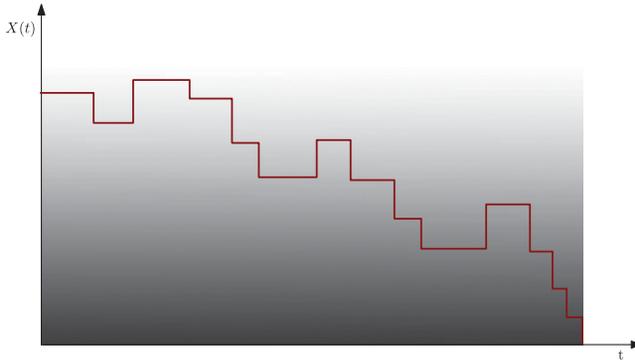
Fig. 2. Possible realization of the birth-death process over a linear positive function.

foraging scenario, the average force exerted on an object, or the average area patrolled by the swarm, can be expressed as the integral of the stochastic process governing the dynamics of the robots.

Let the initial number of active robots be $X(0) = k$ for some $k > 0$ and τ_k denote the extinction time—that is, the first time the birth-death process reaches state 0 starting from state k:

$$\tau_k = \inf\{t > 0 : X(t) = 0 | X(0) = k\}, \tag{3}$$

where the inf of a set S of real numbers is the largest real number that is smaller than or equal to every number in S.

We are interested in integrals of the function f of $X(t)$ from time $t = 0$ to the extinction time τ_k:

$$Y_f(k) = \int_0^{\tau_k} f(X(t))dt. \tag{4}$$

When $f(n) = n$, we will use the notation $Y(k) = Y_f(k)$.

4. EXPECTED VALUE OF THE INTEGRAL OF A LINEAR BIRTH-DEATH PROCESS

In this section, we show how to compute the expected value $\mathbb{E}(Y(k))$ of the integral given in Equation (4). Hernández-Suárez and Castillo-Chavez [1999] propose a basic derivation of $\mathbb{E}(Y(k))$ based on the study of the stationary distribution for different birth-death Markov processes. Their first step in the derivation of $\mathbb{E}(Y(k))$ consists of substituting the transitions to state 0 with transitions to the initial state k (Figure 3). The process $X(t)$ obtained in this way is ergodic and referred to as the modified process. As the modified process is ergodic, when time $t \to \infty$, the state n is visited infinitely often. In the modified process, the starting state is k and the last state of a cycle is state 1. Once state 1 is reached, the cycle is terminated and the modified process jumps again to the initial state k (see Figure 3(b)).

Let us assume that at time t the process has visited state n for r times, and let S_n denote the sum of the random times S_{n_j}, $j = 1, \ldots, r$, the process spent in state n over its r visits to state n:

$$S_n = \sum_{j=1}^{r} S_{n_j}. \tag{5}$$
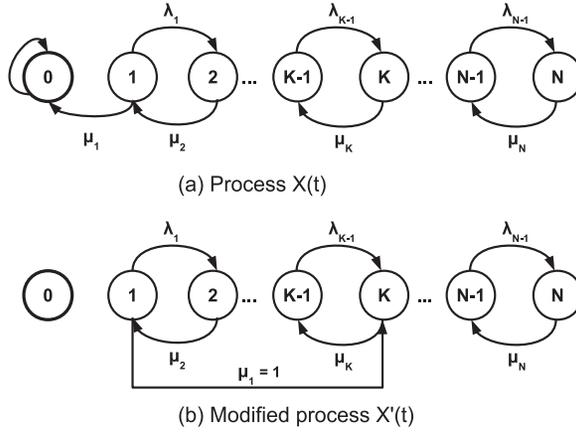
(a) Process X(t)

(b) Modified process X'(t)

Fig. 3. Process $X(t)$ and its modified process $X'(t)$. The modified process is obtained by substituting the transition from state 1 to state 0 with the transition from state 1 to state k. Therefore, state 0 is no longer reachable.

The value of the function $f(n)$ over the r visits to state n when $f(n) = n$ is given by nS_n. Therefore,

$$Y(k) = \int_0^{\tau_k} X(t)dt = \sum_n nS_n. \tag{6}$$

When considering the original process $X(t)$, the expected time $\mathbb{E}(S_n)$ denotes the time $X(t)$ spends in state n before $X(t)$ reaches the absorbing state.

Stefanov and Wang [2000] proposed a method to obtain the more general $\mathbb{E}(Y_f(k))$. Their method relies on a state-space truncation argument. When the integrand function $f(n)$ of the process $X(t)$ is taken into account, similarly to Equation (6), one can write the following:

$$Y_f(k) = \sum_{n=1}^{\infty} f(n)S_n(k), \tag{7}$$

where $S_n(k)$ denotes the total time spent in state n when the initial state is k. We denote by τ_k^m, $Y^m(k)$, and $S^m(k)$ the counterparts of τ_k, $Y(k)$, and $S(k)$, where the state space S^m is truncated at $m \geq k$ (this corresponds to considering a maximum of m elements).

Then we can write the following:

$$\mathbb{E}(Y_f^m(k)) = \mathbb{E}\left(\sum_{n=1}^{m} f(n)S_n^m(k)\right). \tag{8}$$

Using the following result discussed in Stefanov [1995],

$$\mathbb{E}(S_n^m(k)) = \sum_{i=1}^{k} \frac{H_i(n)}{\mu_i}, \tag{9}$$

where

$$H_i(n) = \begin{cases} \frac{\lambda_i \lambda_{i+1} \lambda_{i+2} \ldots \lambda_{n-1}}{\mu_{i+1} \mu_{i+2} \ldots \mu_n} & 1 \leq i < n \\ 1 & i = n \\ 0 & i > n, \end{cases} \tag{10}$$
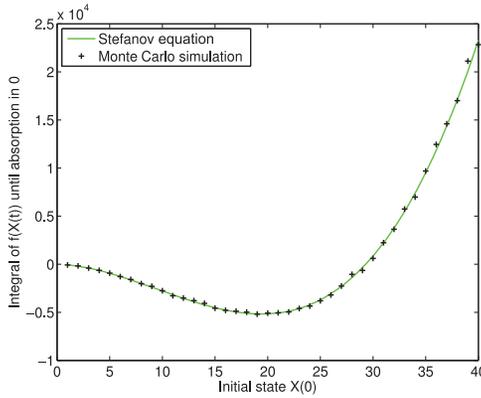
Fig. 4.   Expected value of the integral of a birth-death process with $\lambda = 0.5$ and $\mu = 1$ over the function $f(n) = 2n^2 - 40n$.

with $n \le m$, we can rewrite Equation (8) as follows:

$$\mathbb{E}(Y_f^m(k)) = \mathbb{E}\left(\sum_{n=1}^{m} f(n)S_n^m(k)\right) = \sum_{n=1}^{m} f(n)\mathbb{E}(S_n^m(k)) = \sum_{n=1}^{m} f(n) \sum_{i=1}^{k} \frac{H_i(n)}{\mu_i}$$

$$= \sum_{i=1}^{k} \frac{1}{\mu_i} \sum_{n=i}^{m} f(n)H_i(n). \tag{11}$$

When the state space is infinite, thanks to the monotone convergence theorem [Yeh 2006], Equation (8) can be generalized to the following:

$$\mathbb{E}(Y_f(k)) = \lim_{m \to \infty} \mathbb{E}\left(\sum_{i=1}^{m} f(n)S_n(k)\right), \tag{12}$$

subject to

$$\mathbb{E}\left(\sum_{n=1}^{m} f(n)S_n(k)\right) \le \mathbb{E}(Y_f^m(k)) \le \mathbb{E}(Y_f(k)). \tag{13}$$

The first inequality is due to the smaller state space: as the state space is truncated to $m \ge k$, $S_n^m(k) \geqslant S_n(k)$. The second inequality holds when $n_0 > 0$ exists such that $f(n)$ is nondecreasing for $n \ge n_0$ [Stefanov and Wang 2000] (note that this assumption can be removed by using results of potential theory [Pollett 2003]). In view of Equation (8) and Equation (13), the following proposition holds.

PROPOSITION 4.1 [STEFANOV-WANG].   *The expected value* $\mathbb{E}(Y_f(k))$ *of* $Y_f(k)$ *where the birth-death rates are* λ_i *and* μ_i *has the following closed form expression:*

$$\mathbb{E}(Y_f(k)) = \sum_{i=1}^{k} \left(\frac{1}{\mu_i} \sum_{n=i}^{\infty} f(n)H_i(n)\right). \tag{14}$$

Figure 4 shows the expected value of the integral of a birth-death process with rates $\lambda_i = 0.5$ and $\mu_i = 1$ over the test function $f(n) = 2n^2 - 40n$. The closed form solution is compared to the average obtained with a Monte Carlo simulation over 1,000 runs.
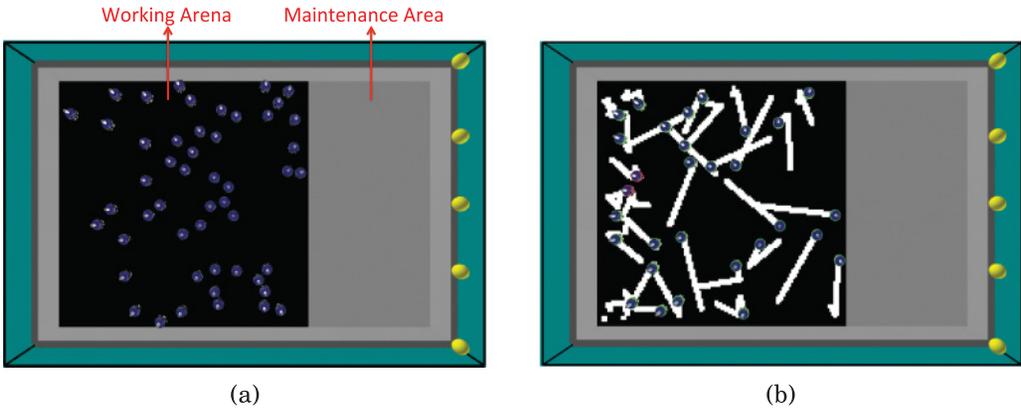
Fig. 5.   Swarm robotics scenario. (a) Initial state: 30 robots are randomly uniformly positioned in the working arena. (b) Snapshot at 35 simulation steps: white trails are areas of the arena that have been cleaned by the robots.

5. AN APPLICATION SCENARIO

In this section, we illustrate how the linear birth-death modeling approach of Equations (1) and (2) can be used to model a robot swarm that performs a cleaning task.[3] We use Equations (4) and (14) to compute several different swarm performance measures: the swarm activity time, the total swarm energy consumption, and the total area cleaned by the swarm during the swarm activity time.

5.1. Cleaning Task

Initially, all robots are active and are uniformly randomly distributed over a 4×4 m^2 closed *working arena* (Figure 5(a)). The robots move randomly in the working arena with a speed of 5cm/s while avoiding collisions with nearby robots.

An active robot can be in either working state or in collision avoidance state. A working robot moves from the working state to the collision avoidance state when spatial interferences occur. While in the collision avoidance state, the robot stops cleaning for whatever time necessary to avoid the collision.

Over time, robots may require maintenance,[4] in which case they move to the maintenance area—2×4 m^2 (see Figure 5(a))—and become inactive. This happens with rate μ for each robot.

When a robot becomes inactive, it invites one of the robots in the maintenance area to become active. This invitation is accepted with a rate $\lambda = \mu\lambda'$, where λ' is a system parameter. Although in theory it could be $\lambda' = 1$, in practice the value of λ' will be lower, as the invited robot might be busy with another task or the message might become lost. Figure 6 illustrates the robot's states and its activation and de-activation dynamics and rates.

5.2. Swarm Performance Measures

In this section, we define the swarm performance measures in which we are interested: the swarm activity time, the total swarm energy consumption, and the total area cleaned by the swarm. These swarm performance measures are computed as the integral over the swarm activity time of the respective swarm robotics functions:

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[3]Instructions for downloading the software are available at: http://iridia.ulb.ac.be/supp/IridiaSupp2015-008/index.html.
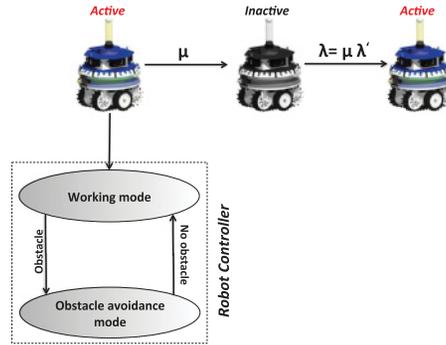[4]For example, because of faults or low battery level.

Fig. 6.   Activation and deactivation dynamics and rates.

(1) *Swarm activity time—the constant case*: $\boldsymbol{f(n) = 1}$. When the swarm robotics function is $f(n) = 1$, the integral of Equation (4) reduces to $Y_f(k) = \int_0^{\tau_k} dt$ and represents the swarm activity time—that is, the time interval from $t = 0$ up to the time $t = \tau_k$—when all robots become inactive (extinction).

(2) *Total swarm energy consumption—the linear case*: $\boldsymbol{f(n) = cn}$. When the swarm robotics function is $f(n) = cn$, where c is a constant, the integral of Equation (4) reduces to $Y_f(k) = \int_0^{\tau_k} cn\,dt$. Under the hypothesis that each robot consumes c energy units per time unit, this integral measures the energy consumed by the swarm during its activity time $[0, \tau_k]$.[5] Note that when $c = 1$, the integral can also represent the total swarm activity time—that is, the sum of the activity times of each robot in the swarm.

(3) *Total area cleaned by the swarm—the nonlinear case*. In the more general case, the swarm robotics function used in Equation (4) is not linear in the number of active robots. Additionally in this case, the integral can be computed using Equation (14). In our cleaning scenario, we are interested in the performance of the swarm measured by the area the robots are able to clean during the swarm activity time. To define the function $f(n)$ that gives the amount of work performed when n robots are active—that is, the area cleaned in all time intervals when exactly n robots were active—we need to take spatial interferences into consideration. However, interferences are a complex phenomenon that depends not only on the physical characteristics of the robots but also on the type of task, on the geometry of the environment, and on the strategy adopted by the robots. Therefore, we use physics-based simulations[6] to estimate the function $f(n)$ for different values of n. To do so, we first run simulations for a selected set of values of n and then use the obtained results to estimate the function $f(n)$.

For each value of $n \in [50, 150]$, we run 30 simulations and compute the average of the total area cleaned by the active robots (Figure 7).

Collision avoidance maneuvers cause performance to decrease when the number of robots increases. This negative effect on the performance of the system due to spatial interferences is shown in Figure 7, in which the average area cleaned by the swarm increases with the number of active robots until the swarm size reaches 93 active robots. Then it starts to decrease because of spatial interferences. The

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[5]Another example of the use of the swarm robotics function $f(n) = cn$ is when c represents the number of seeds dropped per time unit by each robot in an agriculture application. In this case, the integral of Equation (4) represents the total number of seeds dropped by the robot swarm during its activity time.
[6]We use the ARGoS simulator, a state-of-the-art discrete-time physics-based simulation framework that can simulate swarms of robots at different levels of detail [Pinciroli et al. 2012].
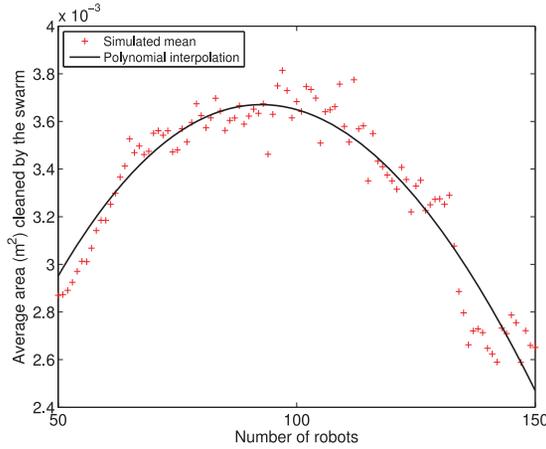
Fig. 7. Average area cleaned by the robot swarm. The red crosses represent the average area cleaned by n active robots in 1s, where n varies between 50 and 150 and the average is obtained via 30 physics-based simulations using the ARGoS simulator. The black solid line is a third-degree polynomial interpolating the red crosses.

swarm robotics function $f(n)$ representing the area cleaned by n active robots is obtained by interpolating the data of Figure 7 with a polynomial of degree 3:

$$f(n) = 0.00000000033583n^3 - 0.00000047651n^2 + 0.000079576n + 0.00012122. \quad (15)$$

5.3. The Model as a Micro-Macro Estimation Tool

In this section, we show how the presented model can be used as a micro-macro estimation tool. In other words, we consider the situation in which the values of the λ and μ parameters of the birth-death process are given, and we show how to use Equation (14) to compute the expected value of the integral of the swarm robotics functions.[7]

We assume that robots are leaving the working arena and stop cleaning with a rate $\mu = 0.1$. The leaving robots send messages to inactive robots so as to activate them. The activation messages succeed in activating robots with a rate $\lambda = 0.5\mu$. This rate is measured using the physics-based simulations performed in ARGoS.

Figure 8 shows the results of a typical simulation run of the cleaning task. Figure 8(a) shows the number of robots that are leaving the cleaning task over time (solid line) and the number of robots that are joining the task to substitute the leaving robots (dashed line)—note that the number of robots joining the task is approximately half the number of robots leaving the task. Figure 8(b) shows the number of active robots over time, starting from time $t = 0$, when the initial number of robots is 100, until the extinction time.

Figure 9(a) through (c) report the three swarm performance measures obtained using Equation (14) and via simulations. All graphs show that there is a very good match between the performance estimated using our modeling approach (i.e., via Equation (14)) and the performance computed using simulations.

5.4. The Model as a Macro-Micro Designing Tool

In this section, we show how the presented model can be used as a design tool when we want the robot swarm to have a certain level of performance on average. This can

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[7]Note that for the constant and linear swarm robotics function cases, it is possible to obtain an exact characterization of their integrals in terms of their probability distribution (see Appendix A).
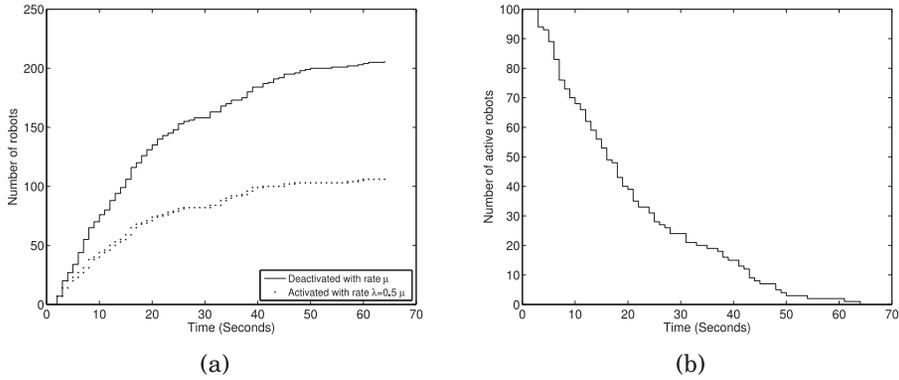
Fig. 8. Typical simulation run of the cleaning task. (a) Number of robots leaving the cleaning task (solid line) and number of robots joining the cleaning task (dashed line). (b) Number of active robots over time.
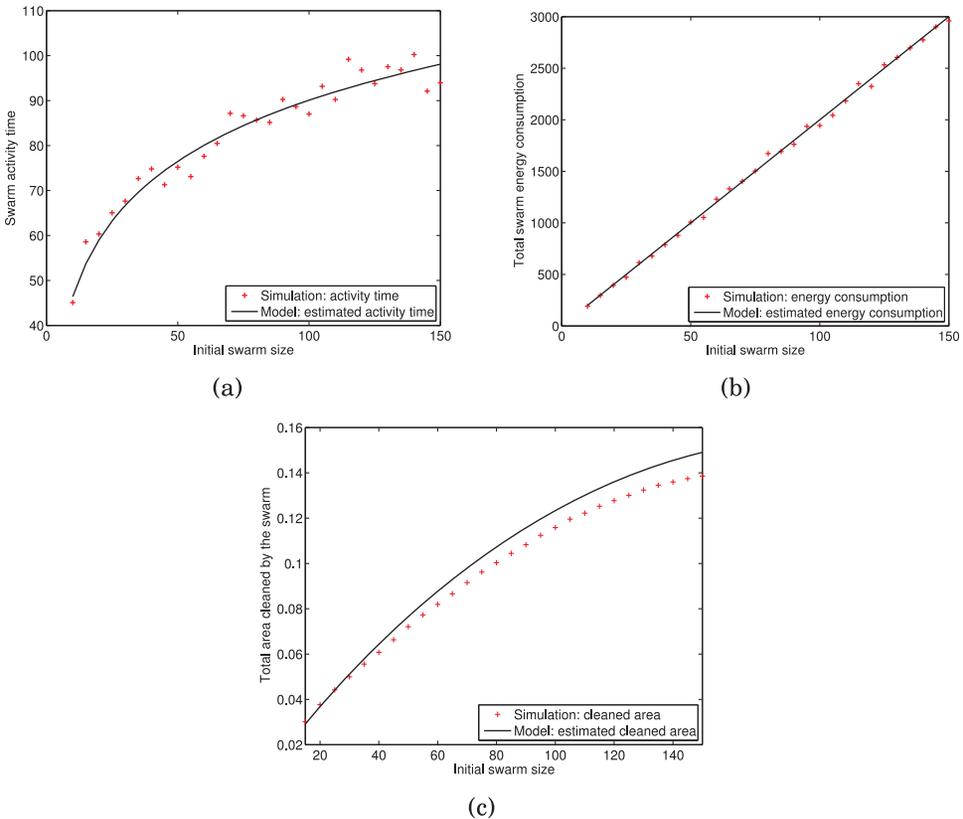
Fig. 9. Comparison between the swarm performance estimated using Equation (14) and computed using simulations. (a) Swarm activity time—simulation results are the average of 100 runs of Monte Carlo simulation. (b) Total swarm energy consumption—simulation results are the average of 100 runs of Monte Carlo simulation. (c) Total area cleaned by the swarm—simulation results are the average of 50 runs of physics-based simulations in ARGoS.
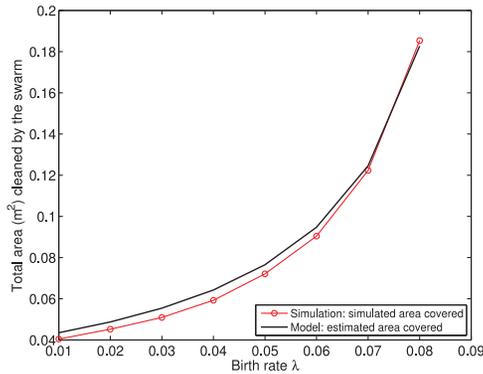
Fig. 10. Expected total area cleaned by the swarm, computed using Equation (14), and the total area cleaned by the swarm obtained as the average of 50 runs of the ARGoS simulator.

be done by using the model to tune the parameters of the birth-death process. Here we consider the case in which the death rate μ is fixed and the birth rate λ is the controllable parameter that we can tune. We use as an example the nonlinear swarm robotics function that represents the total area cleaned by the robots during the activity time of the swarm. Figure 10 shows the area of the working arena that was cleaned by a swarm of initial size $n = 50$ robots and with a varying birth rate $\lambda \in [0.1\mu - 0.8\mu]$. The graph shows that when the birth rate increases, the amount of work performed during the swarm activity time increases. A user can use the results of Figure 10 to select a birth rate for which the robot swarm has the required average performance.

Another parameter that can be tuned to influence the performance of the system is the initial number of robots assigned to the task. Figure 9(c) shows how changing this parameter can influence the cleaned area while keeping the birth and death rates fixed.

6. CONCLUSIONS

In this article, we investigated the use of linear birth-death processes for the probabilistic modeling of swarm robotics systems. In many swarm robotics applications, the number of robots in a particular state or engaged in a particular task constitutes the main descriptor of the state of the swarm. We have shown that when the evolution in time of this number can be modeled using a linear birth-death process, well-established results can be applied to characterize the integral of swarm robotics functions that describe aspects of the swarm activity that we want to measure. These measures include, for example, the expected total swarm energy consumption or the expected total amount of work performed by the swarm.

In this context, we reviewed theoretical results for the evaluation of the expected value of the integral of swarm robotics functions for linear birth-death processes and provided references to relevant literature. To illustrate these approaches, we considered examples in which the dynamics of the activation and deactivation of robots can be modeled as a linear birth-death process, and we showed that our modeling approach can be used both to predict the robot swarm performance and as a tool to design a robot swarm that has an expected performance.

Future work will consider the use of the results presented in this article to guide the development of individual robotics controllers so as to obtain swarm level specific performance guarantees.

APPENDIX A: EXACT CHARACTERIZATION OF THE INTEGRAL OF SWARM ROBOTICS FUNCTION

In this appendix, we briefly show how to compute the exact value of Equation (4) for the particular cases of swarm robotics functions $f(n) = 1$ and $f(n) = cn$.

A1: The $f(n) = 1$ Case

The probability density function of $\int_0^{\tau_k} dt$ can be computed using passage time probabilities. In fact, for linear birth-death processes, closed-form solutions for the passage time probabilities $P_{m,n}(t) = Pr(X(t) = n | X(0) = m)$ are available [Bailey 1990]:

$$P_{m,n}(t) = \sum_{j=0}^{\min(m,n)} \binom{m}{j} \binom{m+n-j-1}{m-1} \frac{\alpha^{m-j}}{\beta^{-n+j}} (1 - \alpha - \beta)^j, \qquad (16)$$

where

$$\alpha = \frac{\mu(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu} \quad \text{and} \quad \beta = \frac{\lambda(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu}.$$

The absorption probability as a function of time, from an initial state m to the absorbing state 0, is [Bailey 1990]:

$$P_{m,0}(t) = \left(\frac{\mu(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu} \right)^m. \qquad (17)$$

The probability density function of the swarm activity time is obtained from Equation (17) [Bailey 1990]:

$$f_{m,0}(t) = \frac{m\mu(e^{(\lambda-\mu)t} - 1)^{m-1}\mu(\lambda - \mu)^2 e^{(\lambda-\mu)t}}{(\lambda e^{(\lambda-\mu)t} - \mu)^{m+1}}. \qquad (18)$$

Figure 11 shows the probability density function of the swarm activity time computed using Equation (18), as well as numerical results obtained via 1,000 Monte Carlo simulations.
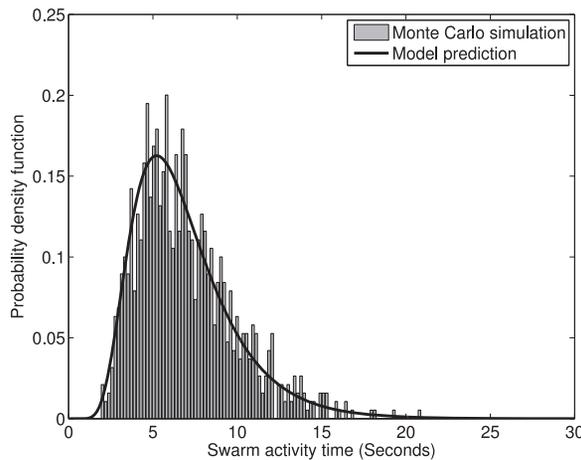

Fig. 11. Probability density function of the total swarm activity time when the swarm is modeled as a linear birth-death process. Solid line: as computed with Equation (18); histogram: 1,000 Monte Carlo simulations. Initial number of active robots = 20, $\lambda = 0.6$, $\mu = 1$.
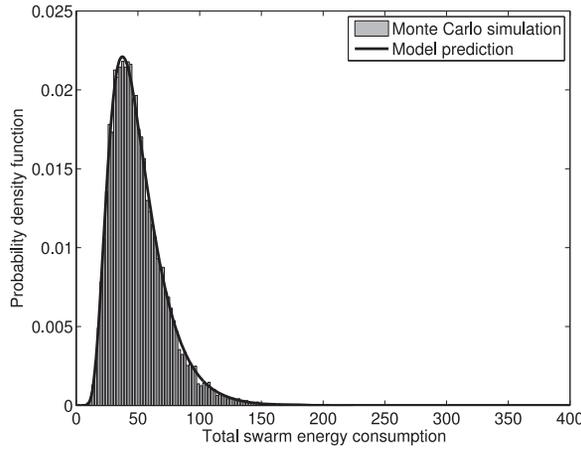
Fig. 12. Probability density function of the total swarm energy consumption when the swarm is modeled as a linear birth-death process. Solid line: as computed with Equation (19); histogram: $1,000$ Monte Carlo simulations. Initial number of robots $= 20$, $\lambda = 0.6$, $\mu = 1$, $c = 1$.
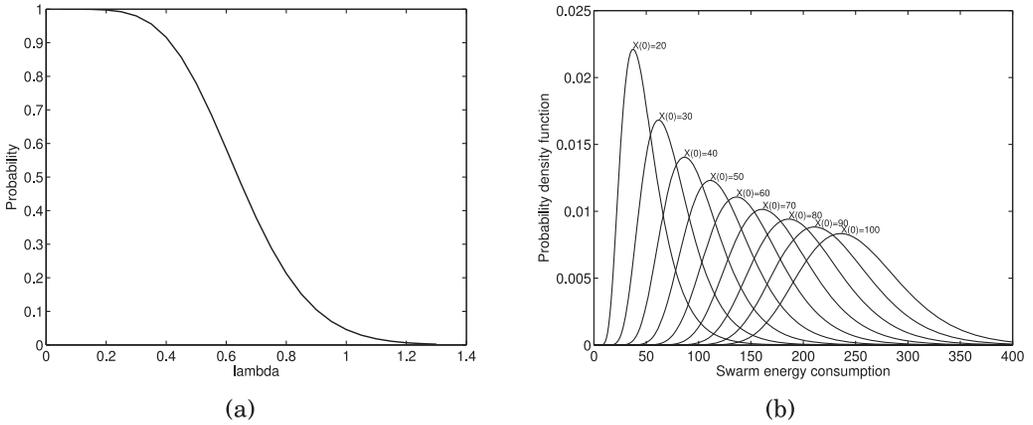

(a)   (b)

Fig. 13. (a) Probability that the total swarm energy consumption is lower than 50 energy units as a function of the activation rate λ ($\mu = 1$). (b) Probability density functions of the total swarm energy consumption for different initial swarm sizes.

A2: The $f(n) = cn$ Case

The probability density function of $\int_0^{\tau_k} cn\,dt$ can be computed using the inverted Laplace transform [Pollett and Stefanov 2002]. In the case of linear birth-death processes and of energy consumption per time unit $c = 1$, the inverted Laplace transform is given by Pollett [2003]:

$$dP_n(\mathcal{E} \leq x) = \frac{n}{x} e^{-(\lambda+\mu)x}(\mu/\lambda)^{n/2} I_n(2x\sqrt{\lambda\mu})dx, \tag{19}$$

where $\mathcal{E} = \int_0^{\tau_k} n\,dt$ and $I_n(z)$ is the modified Bessel function of the first kind.

Figure 12 shows the probability density function of the total swarm energy consumption. Equation (19) can be employed for the determination of the parameters that guarantee a total swarm energy consumption under a given critical level with a certain probability. For example, Figure 13(a) shows the probability as a function of the activation rate λ that the total energy consumed by the swarm is lower than 50 energy units. Different characteristics concerning the swarm energy consumption, such as the range

of this consumption, can be obtained a priori for any initial swarm size. Figure 13(b) reports the probability density function of the total swarm energy consumption for different initial numbers of active robots.

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